SOME EXACT SOLUTIONS OF THE DIRAC EQUATION

A.S. DE CASTRO
DFQ, UNESP, C.P. 205, 12500-000 Guaratinguetá SP, Brasil
E-mail: castro@feg.unesp.br

J. FRANKLIN
Department of Physics, Temple University, Philadelphia, PA
E-mail: v5030e@vm.temple.edu

Exact analytic solutions are found to the Dirac equation for a combination of Lorentz scalar and vector Coulombic potentials with additional non-Coulombic parts. An appropriate linear combination of Lorentz scalar and vector non-Coulombic potentials, with the scalar part dominating, can be chosen to give exact analytic Dirac wave functions.

In a previous letter,[1] simple exact solutions were found for the Dirac equation for the combination of a Lorentz vector Coulomb potential with a linear confining potential that was a particular combination of Lorentz scalar and vector parts. In this work, we extend the method of Ref. 1 to the more general case of an arbitrary combination of Lorentz scalar and vector Coulombic potentials with a particular combination of Lorentz scalar and vector non-Coulombic potentials. A more complete version of this work will appear elsewhere.[2]

The Dirac equation we solve is

$$\left[\alpha \cdot \mathbf{p} + \beta m - \frac{\lambda + \beta\eta}{r} + V(r) + \beta S(r)\right] \psi = E\psi, \tag{1}$$

where $\boldsymbol{\alpha}$ and β are the usual Dirac matrices. The four component wave function ψ can be written in terms of two component spinors u (upper component) and v (lower component) satisfying the equations

$$(\sigma \cdot \mathbf{p})v = \left[E - m + \frac{\lambda + \eta}{r} - V(r) - S(r)\right] u \tag{2}$$

$$(\sigma \cdot \mathbf{p})u = \left[E + m + \frac{\lambda - \eta}{r} - V(r) + S(r)\right] v. \tag{3}$$

The key step in generating relatively simple exact solutions of the Dirac equation is to choose a particularly simple form for the function $v(r)$. For s-states,[a] we choose $v(r) = i\gamma(\sigma \cdot \hat{\mathbf{r}})u(r)$, where γ is a constant factor to be determined

[a]States with orbital angular momentum are treated in Ref. 2.

by the solution to the Dirac equation. This is the form of $v(r)$ that was found in Ref. 1 for a Coulomb plus linear confining potential. This ansatz for $v(r)$ has also been used as the basis for generating approximate saddle point solutions for the Dirac[3] and Breit[4] equations. Using this form of $v(r)$, equations (2) and (3) reduce to two first order ordinary diffential equations for $u(r)$

$$\frac{du}{dr} = \frac{1}{\gamma}\left[E - m - V(r) - S(r) + \frac{\lambda + \eta - 2\gamma}{r}\right] u(r) \tag{4}$$

$$\frac{du}{dr} = -\gamma\left[E + m - V(r) + S(r) + \frac{\lambda - \eta}{r}\right] u(r). \tag{5}$$

Equations (4) and (5) are two independent equations for the same quantity, so that each term in one equation can be equated with the corresponding term in the other equation having the same radial dependence. This leads to

$$\gamma^2 = \frac{m - E}{m + E} = \frac{S(r) + V(r)}{S(r) - V(r)} = \frac{\lambda + \eta - 2\gamma}{\eta - \lambda}. \tag{6}$$

The relations in Eq. (6) can be rearranged, after some algebra, to give

$$\gamma = \frac{\lambda + \eta}{1 + b}, \quad b = \pm\sqrt{1 - \lambda^2 + \eta^2} \tag{7}$$

The constant b can have either sign. Although b must be positive in the pure Coulombic case, we will see that a negative b is possible if the Lorentz scalar potential $S(r)$ is more singular at the origin than $1/r$. The bound state energy can be written as

$$E = m\left(\frac{1 - \gamma^2}{1 + \gamma^2}\right) = m\left(\frac{b\lambda - \eta}{\lambda - b\eta}\right) = -m\frac{V(r)}{S(r)}. \tag{8}$$

The wave function $u(r)$ can be found by solving Eq. (5) to give

$$u(r) = r^{b-1} \exp\left[-a\left(r + \frac{1}{m}\int S(r)dr\right)\right], \tag{9}$$

where the constant a is given by $a = \gamma(m + E) = \pm\sqrt{m^2 - E^2}$. The constants γ and a can have either sign if $S(r)$ approaches a constant or diverges as r becomes infinite. The integral in Eq. (9) can diverge for any finite r, as long as the product of a times the integral diverges in the positive sense. The integral can also diverge at the origin or as $r \to \infty$ as long as the quantity in square brackets in Eq. (9) remains negative.

The last equality in Eq. (8) shows that in order for this class of exact solutions to apply, the Lorentz vector and scalar non-Coulombic potentials must have the same radial dependence and opposite sign, with the vector potential being smaller than the scalar potential. As long as this constraint is satisfied, the results in Eqs. (7)-(9) represent a complete exact solution for the ground state wave function and energy of the Dirac Hamiltonian given in equation (1).

Because of the constraints imposed on the potentials by Eq. (8), the energy can be written purely in terms of one set of potentials or the other. The ground state energy can be specified by the Coulombic constants λ and η, with the other equality in Eq. (8) serving as a constraint on the non-Coulombic potentials $V(r)$ and $S(r)$. Or the ground state energy could be specified by the ratio $V(r)/S(r)$ of the non-Coulombic potentials with the other equality serving as a constraint on λ and η. Although the possibility of this class of exact solutions is limited by the constraints on the potentials, this still permits a wide range of non-Coulombic potentials.

We now consider conditions imposed on the potentials and the wave function parameters by the physical requirements that the potentials be real and the wave function normalizable. We see from Eq. (6) that γ must be real, and then from Eq. (7) that b must be real. This requires the Coulombic potentials to satisfy the condition $1 - \lambda^2 + \eta^2 \geq 0$. The reality of γ restricts possible bound state energies to the range $-m < E < m$. Note that negative energies can occur, but $E + m$ cannot be negative. Also, E cannot equal $\pm m$, because this would make $\gamma = 0$, leading to a constant, unormalizable wave function. This condition on E, along with Eq. (9), means that $V(r)$ must always be less in magnitude than $S(r)$.

We discuss the remaining conditions on the parameters in terms of three sub-classes of solution:

1. The "normal" class of solutions has b, γ, and a all positive. In this case, we see from Eq. (7) that the Coulombic potentials must satisfy the further condition $\lambda + \eta > 0$.

2. This sub-case has b negative, with γ and a still positive. The constant b can be negative if the product $aS(r)$ is positively divergent at the origin faster than $1/r$. Then each of Eqs. (7)-(10) hold just as for positive b, and the wave function is still normalizable. The states with positive and negative b are not two ground states of the same Hamiltonian because the potentials cannot be the same for each state. That is, either the Coulombic potentials or the non-Coulombic potentials must change to be consistent with a negative b. Sub-case 1 with positive b transforms smoothly into the pure Coulombic solution as the non-Coulombic potential tends to zero everywhere. But this is not true for sub-

case 2 with b negative. This sub-case requires the non-Coulombic potential to be dominant at the origin, and so has no corresponding pure Coulombic limit.

3. This sub-case has a negative γ and a negative a, while b can have either sign, as discussed in sub-cases 1 and 2 above. A negative a is possible if the non-Coulombic potential diverges or approaches a constant as $r \to \infty$, so that the integral in Eq. (9) diverges faster than r at large r. Since a is negative, the potential $S(r)$ must be *negative* at large r. Then all of Eqs. (7)-(10) hold as for positive a, and the wave function is still normalizable. This case is highly unusual, because it allows the possibility of a potential that is negative everywhere and diverges negatively at both the origin and infinite r. We know of no other example in quantum mechanics where such a negative potential can lead to a normalizable ground state. The reason this is possible here can be seen from Eqs. (4) and (5). There it is seen that $S(r)$ enters the differential equations for $u(r)$ only in the combinations $\gamma S(r)$ or $S(r)/\gamma$. Since these effective potentials are positive, the resulting wave function is normalizable. As with sub-case 2, the case with γ and a negative does not approach a pure Coulombic case if the non-Coulombic potential tends to zero.

We now look at some special cases. If the non-Coulombic potentials are absent, then the solutions are for a general linear combination of Lorentz vector and Lorentz scalar Coulombic potentials. If either constant, λ or η, is zero, we recover the usual solutions of the Dirac equation for a pure scalar or vector Coulombic potential. The Coulombic potentials cannot both be absent (while keeping a non-Coulombic part) because then γ would be zero leading to a constant, unnormalizable wave function.

The method we have described does not work for radially excited states, because the simple ansatz for $v(r)$ does not lead to consistent equations for du/dr in that case. The method does work for the lowest orbitally excited state for which $l = j - \frac{1}{2}$. That case is discussed in Ref. 2.

Acknowledgments

One of the authors (A.S.C.) would like thank FAPESP for financial support.

References

1. J. Franklin, Mod. Phys. Lett. A **14**, 2409 (1999).
2. A. S. De Castro and J. Franklin, Int. J. Mod. Phys. A **15**, 4355 (2000).
3. J. Franklin and R. Intemann, Phys. Rev. Lett. **54**, 2068 (1985);
   J. Franklin and J. Giammarco, Phys. Rev. A **36**, 5839 (1987).
4. J. Giammarco and J. Franklin, Nucl. Phys. A **585**, 450 (1995).